XI.—*On an Extension of a Theorem of Euler, with a Determination of the Limit beyond which it fails.* *By* John Radford Young, Professor of Mathematics in Belfast College.

Read November 30, 1847.

THE theorem, into the extension and necessary limitation of which it is here proposed to examine, was first demonstrated by Euler, under the following enunciation:—The sum of four squares multiplied by the sum of four squares gives a product which is also the sum of four squares.

This theorem has been lately called in request by Sir William R. Hamilton, in connexion with his researches in reference to the calculus of quaternions; in the establishment of the fundamental principles of which he was independently led to the theorem in question, as a consequence and a confirmation of the consistency of his original conventions, with respect to the new imaginary quantities which that calculus involves.

Attention thus came to be directed to the inquiry, as to whether or not Euler's theorem admits of extension; for it was naturally enough suspected that if such extension should be found to be practicable, a corresponding extension of the new theory of imaginaries would suggest itself.

Mr. John T. Graves, who, as well as Professors Charles Graves and De Morgan, has paid much attention to these researches, was, I believe, the first to announce that the theorem in question was equally applicable to sums of *eight* squares; and he remarks that, "as Euler's theorem is connected with Hamilton's quaternions, so my theorem concerning sums of eight squares may be made the basis of *sets of eight*, and was actually so applied by me, about Christmas, 1843. But," he adds, "the full statement and proof of the theorem concerning sums of eight squares, and of several other new theorems connected with the doctrine of numbers, must be reserved for another time."

This quotation is from the Philosophical Magazine for April, 1845. No demonstration, however, of the extended case of Euler's theorem having appeared, the author of the present communication was led to examine into the practicability of this extension; and, early in May last, arrived at the conclusion that, as previously affirmed by Mr. J. T. Graves, the theorem really had place for sums of eight squares, as well as for sums of two and of four. He communicated his investigation to the Royal Irish Academy in the beginning of June; and the formula for eight squares, to which this investigation conducted, was printed in the "Proceedings" for the then current month. At the same time Mr. Graves's results were also presented: both formulæ appear on the same page of the "Proceedings;" and, as they are equally correct, it is needless to add that they are mutually convertible into each other.

In drawing up this more detailed communication for the Transactions, the writer should not have thought it worth while, in a matter of such comparative insignificance, to have adverted to these preliminary particulars, but for his anxiety to avoid all appearance of preferring claims, however unimportant, to which he has no title. He may, perhaps, be permitted to add, that, even up to this date, no investigation of the principles upon which the eight-square theorem depends has been made public; nor has any intimation been as yet given, as to whether the proposition terminates here, or may be extended indefinitely. It was the author's original impression that such an indefinite extension was really practicable; and such, he has reason to believe, was the prevailing opinion with those to whom Mr. Graves's results had been made known. It is probable, therefore, that the principal point of interest—perhaps the only point of interest—which the present paper offers, is that which sets all conjecture on this head at rest, by showing that the supposed extension is impossible.

To these introductory remarks it may not be superfluous to add, that the demonstration hereafter given, of the impossibility of the sixteen-square form, except under special restrictions, adds one more to the very few instances hitherto furnished of *proving a negative:* that is, of showing that a proposition, the affirmative of which would fully harmonize with pre-established truths, and actually involve them in its expression, is, nevertheless, impossible. In modern times, the most remarkable specimen of a proof of this kind is that which has been offered by Abel and Hamilton, in their researches respecting general for-

mulæ for the solution of equations of the fifth degree. And it seems to be not undeserving of a passing notice that, as regards this latter important inquiry, as well as in reference to the more humble researches of this paper, ordinary arithmetic is found adequate to accomplish what is inherently impossible in algebraic symbols, whenever, by giving to those symbols a numerical interpretation, they are brought under the laws and operations of that science. All equations with numerical coefficients and numerical roots, may be solved by processes purely arithmetical; and, in like manner, the sixteen-square integral formula is always attainable, when the proposed factors are themselves composed of integral square numbers. This will at once appear from considering that, whatever be the product of these two sets of sixteen, we may separate it into four numerical parcels; and that each of these may be replaced by the sum of four squares.* Writings devoted to the theory of numbers are, in general, as much occupied with the relations, identities, decompositions, &c., of *algebraical* expressions as with those of pure *numbers;* and I cannot help thinking that greater explicitness and precision would be given to such writings, if the terms *algebraic integer* and *numerical integer* were employed in their proper distinctive senses. A similar remark applies to the term *function*, which, in some cases, may mean an algebraic or a purely symbolical function; and, in other cases, a strictly numerical function. The roots of an equation are functions of its coefficients, and, *therefore*, it has been inferred that the former are *algebraically* expressible in terms of the latter. The researches above adverted to have proved that this inference has been hastily made; yet the *numerical* expressions for the roots, by aid of the coefficients,—whenever the case is brought within the limits of arithmetic,—are always attainable.

These preliminary observations will, I trust, not be considered as irrelevant; for I am anxious that it should be understood, throughout the following investigations, that the relations and identities discussed are purely algebraic, and thus admit of the widest interpretation; and that when a property or relation is said not to have place, or to fail, it is to be understood that the algebraic conditions are not fulfilled in all their generality; and not that the analogous property may not hold in arithmetical integers. The integers spoken of here refer always to algebraic forms, and, of course, include those of arithmetic.

* Legendre, Théorie des Nombres, p. 202.

In instituting an inquiry into the practicability of generalizing the above-mentioned theorem of Euler, the first thing that presents itself to the mind is, that, if the supposed generalization of the theorem existed, it is reasonable to expect that something like a uniform law would be observed to prevail among the component terms of the products in the known particular cases; which law might serve to suggest the anticipated extension. Let us, then, examine into the constitution of these products, in the established forms for two and for four squares. These products are respectively

$$(y'^2+z'^2)(y^2+z^2)=(yy'+zz')^2+(yz'-zy')^2,$$

and

$$\begin{aligned}(w'^2+x'^2+y'^2+z'^2)(w^2+x^2+y^2+z^2)=\ &(ww'+xx'+yy'+zz')^2+\\ &(wx'-xw'+yz'-zy')^2+\\ &(wy'-yw'+zx'-xz')^2+\\ &(wz'-zw'+xy'-yx')^2.\end{aligned}$$

In the first of these cases we may observe, that the roots of the resulting squares arise from multiplying the roots of the original squares, as follows:

$$\begin{array}{l} y'+z' \\ \underline{y\ +z} \\ yy'+zz' \\ yz'-zy' \text{ or } zy'-yz', \end{array}$$

the signs being made alternate in the second row. In like manner, as respects the resulting squares in the other case, the roots of these also are constructed from those of the original squares, in a way so analogous as to furnish indications of a general principle ; thus,

$$\begin{array}{l} w'+x'+y'+z' \\ \underline{w\ +x\ +y\ +z} \\ ww'+xx'+yy'+zz' \\ wx'-xw'+yz'-zy' \\ wy'-yw'+zx'-xz' \\ wz'-zw'+xy'-yx', \end{array}$$

in which construction it will be perceived that, guided by the operation in the first case, we commence with the symmetrically situated pairs,

$$w' + x',\ w + x, \text{ and } y' + z',\ y + z\,;$$

then proceed to the symmetrically placed pairs,

$$w' + y',\ w + y, \text{ and } x' + z',\ x + z\,;$$

and finally to the remaining symmetrical pairs,

$$w' + z',\ w + z, \text{ and } x' + y',\ x + y\,;$$

it being observed, in reference to the last four of these pairs, that the first row of results are exhibited in the leading row of the complete form.

Still submitting to the guidance of the same principle of construction, and proceeding to the case of eight squares, the following results present themselves:

$$\begin{array}{llllllll|l}
s' & + t' & + u' & + v' & + w' & + x' & + y' & + z' & \\
s & + t & + u & + v & + w & + x & + y & + z & \\
\hline
ss' & + tt' & + uu' & + vv' & + ww' & + xx' & + yy' & + zz' & (s'') \\
st' & - ts' & + uv' & - vu' & + wx' & - xw' & + yz' & - zy' & (t'') \\
su' & - us' & + vt' & - tv' & + yw' & - wy' & + xz' & - zx' & (u'') \\
sv' & - vs' & + tu' & - ut' & + wz' & - zw' & + xy' & - yx' & (v'') \\
sw' & - ws' & + xt' & - tx' & + uy' & - yu' & + zv' & - vz' & (w'') \\
sx' & - xs' & + tw' & - wt' & + yv' & - vy' & + zu' & - uz' & (x'') \\
sy' & - ys' & + zt' & - tz' & + wu' & - uw' & + vx' & - xv' & (y'') \\
sz' & - zs' & + ty' & - yt' & + vw' & - wv' & + ux' & - xu'. & (z'')
\end{array}$$

Now, just as, in the former case, we commenced the rows in order with $w + x$, $w + y$, and $w + z$, in succession, so here we begin with $s + t + u + v$, $s + t + w + x$, and $s + t + y + z$, in succession.

When these several sets of four are disposed of, agreeably to the above general form for fours, then the set of four which remains, after the imagined suppression of these, is at each step to be treated in the same way. And whether we are to take the several rows of these second sets of results with the same signs that they have in the pre-established form for four, or with all the signs in any one of those rows changed,—which is, of course, allowable, without any violation of that form,—is to be determined by this circumstance, viz.: we are to take care that the two consecutive binomials, or pairs of results, in each row for eight, all conform to the general model for four. Attending to this arrangement, the consecutive pairs of binomials in any row, as in the fifth, for example,

$$\begin{aligned} &sw' - ws' + xt' - tx' \\ &xt' - tx' + uy' - yu' \\ &uy' - yu' + zv' - vz', \end{aligned}$$

all conform to the previously established model for four, and harmonize with the other binomials in which the same quantities occur. At present, this arrangement may be regarded simply as matter of observation: there is, indeed, a conformity to the model still more general than this, and which we shall find to be equally essential to the accuracy of the eight-square formula; it will be more distinctly adverted to presently: but as the additional conformity here hinted at is a spontaneous result of that stipulated for above, no cautionary precepts, beyond those now mentioned, are necessary for the actual construction of the eight-square formula.

It was at this stage of the investigation that the writer of the present communication, abandoning all further attempts at practically applying the principles that had thus led him to correct constructions of the four-square and eight-square formulæ, resorted to theoretical considerations and abstract general reasonings, to prove that the laws of construction, thus far found to be successful, must necessarily prevail in the sixteen-square and subsequent forms. He conceived that when the sixteen rows of combinations,—each dashed quantity being combined with an undashed one, as in the preceding cases,—were written down without any intervening signs, and which would involve little or no trouble:—he conceived that it was always *possible* afterwards to introduce such an arrangement of signs as to bring about the conformity here alluded to. And it was not till after the expenditure of much time, and the trial of every arrangement that gave any promise of success in the case of sixteen squares, that he was led to suspect the applicability of the assumed principle beyond the eight-square form already attained; and thence to the demonstration of its utter failure beyond that limit.

Before entering upon this demonstration, it may be well to verify, by actual development, the expression for the product of eight squares already given; not that such development is absolutely necessary for the purpose of verification, for, as will shortly be noticed, abundant proof of the accuracy of the formula may be derived from a simple inspection of its component terms; but it will be found convenient, in showing the impossibility of the sixteen-square form, to

have the form for eight squares already developed in full. The following seems to be the easiest way of writing out the developments in question:

$$
\begin{aligned}
&(ss' + tt' + uu' + vv' + ww' + xx' + yy' + zz')^2 = \\
&\quad (ss')^2 + 2ss'(tt' + uu' + vv' + ww' + xx' + yy' + zz') \\
&+ (tt')^2 + 2tt'(uu' + vv' + ww' + xx' + yy' + zz') \\
&+ (uu')^2 + 2uu'(vv' + ww' + xx' + yy' + zz') \\
&+ (vv')^2 + 2vv'(ww' + xx' + yy' + zz') \\
&+ (ww')^2 + 2ww'(xx' + yy' + zz') \\
&+ (xx')^2 + 2xx'(yy' + zz') \\
&+ (yy')^2 + 2yy'(zz') \\
&+ (zz')^2.
\end{aligned}
$$

$$
\begin{aligned}
&(st' - ts' + uv' - vu' + wx' - xw' + yz' - zy')^2 = \\
&\quad (st')^2 + 2st'(- ts' + uv' - vu' + wx' - xw' + yz' - zy') \\
&+ (ts')^2 - 2ts'(uv' - vu' + wx' - xw' + yz' - zy') \\
&+ (uv')^2 + 2uv'(- vu' + wx' - xw' + yz' - zy') \\
&+ (vu')^2 - 2vu'(wx' - xw' + yz' - zy') \\
&+ (wx')^2 + 2wx'(- xw' + yz' - zy') \\
&+ (xw')^2 - 2xw'(yz' - zy') \\
&+ (yz')^2 + 2yz'(- zy') \\
&+ (zy')^2.
\end{aligned}
$$

$$
\begin{aligned}
&(su' - us' + vt' - tv' + yw' - wy' + xz' - zx')^2 = \\
&\quad (su')^2 + 2su'(- us' + vt' - tv' + yw' - wy' + xz' - zx') \\
&+ (us')^2 - 2us'(vt' - tv' + yw' - wy' + xz' - zx') \\
&+ (vt')^2 + 2vt'(- tv' + yw' - wy' + xz' - zx') \\
&+ (tv')^2 - 2tv'(yw' - wy' + xz' - zx') \\
&+ (yw')^2 + 2yw'(- wy' + xz' - zx') \\
&+ (wy')^2 - 2wy'(xz' - zx') \\
&+ (xz')^2 + 2xz'(- zx') \\
&+ (zx')^2.
\end{aligned}
$$

$$
\begin{aligned}
&(sv' - vs' + tu' - ut' + wz' - zw' + xy' - yx')^2 = \\
&(sv')^2 + 2sv'(-vs' + tu' - ut' + wz' - zw' + xy' - yx') \\
+&(vs')^2 - 2vs'(tu' - ut' + wz' - zw' + xy' - yx') \\
+&(tu')^2 + 2tu'(-ut' + wz' - zw' + xy' - yx') \\
+&(ut')^2 - 2ut'(wz' - zw' + xy' - yx') \\
+&(wz')^2 + 2wz'(-zw' + xy' - yx') \\
+&(zw')^2 - 2zw'(xy' - yx') \\
+&(xy')^2 + 2xy'(-yx') \\
+&(yx')^2.
\end{aligned}
$$

$$
\begin{aligned}
&(sw' - ws' + xt' - tx' + uy' - yu' + zv' - vz')^2 = \\
&(sw')^2 + 2sw'(-ws' + xt' - tx' + uy' - yu' + zv' - vz') \\
+&(ws')^2 - 2ws'(xt' - tx' + uy' - yu' + zv' - vz') \\
+&(xt')^2 + 2xt'(-tx' + uy' - yu' + zv' - vz') \\
+&(tx')^2 - 2tx'(uy' - yu' + zv' - vz') \\
+&(uy')^2 + 2uy'(-yu' + zv' - vz') \\
+&(yu')^2 - 2yu'(zv' - vz') \\
+&(zv')^2 + 2zv'(-vz') \\
+&(vz')^2.
\end{aligned}
$$

$$
\begin{aligned}
&(sx' - xs' + tw' - wt' + yv' - vy' + zu' - uz')^2 = \\
&(sx')^2 + 2sx'(-xs' + tw' - wt' + yv' - vy' + zu' - uz') \\
+&(xs')^2 - 2xs'(tw' - wt' + yv' - vy' + zu' - uz') \\
+&(tw')^2 + 2tw'(-wt' + yv' - vy' + zu' - uz') \\
+&(wt')^2 - 2wt'(yv' - vy' + zu' - uz') \\
+&(yv')^2 + 2yv'(-vy' + zu' - uz') \\
+&(vy')^2 - 2vy'(zu' - uz') \\
+&(zu')^2 + 2zu'(-uz') \\
+&(uz')^2.
\end{aligned}
$$

$$(sy' - ys' + zt' - tz' + wu' - uw' + vx' - xv')^2 =$$
$$(sy')^2 + 2sy'(-ys' + zt' - tz' + wu' - uw' + vx' - xv')$$
$$+ (ys')^2 - 2ys'(zt' - tz' + wu' - uw' + vx' - xv')$$
$$+ (zt')^2 + 2zt'(-tz' + wu' - uw' + vx' - xv')$$
$$+ (tz')^2 - 2tz'(wu' - uw' + vx' - xv')$$
$$+ (wu')^2 + 2wu'(-uw' + vx' - xv')$$
$$+ (uw')^2 - 2uw'(vx' - xv')$$
$$+ (vx')^2 + 2vx'(-xv')$$
$$+ (xv')^2.$$

$$(sz' - zs' + ty' - yt' + vw' - wv' + ux' - xu')^2 =$$
$$(sz')^2 + 2sz'(-zs' + ty' - yt' + vw' - wv' + ux' - xu')$$
$$+ (zs')^2 - 2zs'(ty' - yt' + vw' - wv' + ux' - xu')$$
$$+ (ty')^2 + 2ty'(-yt' + vw' - wv' + ux' - xu')$$
$$+ (yt')^2 - 2yt'(vw' - wv' + ux' - xu')$$
$$+ (vw')^2 + 2vw'(-wv' + ux' - xu')$$
$$+ (wv')^2 - 2wv'(ux' - xu')$$
$$+ (ux')^2 + 2ux'(-xu')$$
$$+ (xu')^2.$$

Now, the double products in all these groups will be found to cancel. We may readily satisfy ourselves of this as follows. Commencing with the first set of double products, let there be written beneath the successive terms within the vincula the numbers 1, 2, 3, &c., in order, up to the number 28, which will fall beneath the term zz', in the last vinculum: then, upon searching among the other groups, twenty-eight terms will be found to cancel these. Now, let the gaps in the second group be filled up by continuing the numbers inserted in the first, seeking, however, at every new insertion, for the proper neutralizing term among the succeeding groups; under which term, when found, the same number is to be written. In this way, continuing to fill up the chasms in the several groups, one after another, all the cancelling terms may easily be disco-

vered; and, by having thus marked each pair with the same number, the means of readily revising the comparisons will be secured. The result will be, that all the terms, except the vertical rows of squares, will disappear from the developments; and it is obvious that these squares, sixty-four in number, constitute the product of those originally proposed: therefore,

$$\begin{aligned}&(s'^2 + t'^2 + u'^2 + v'^2 + w'^2 + x'^2 + y'^2 + z'^2) \times \\ &(s^2 + t^2 + u^2 + v^2 + w^2 + x^2 + y^2 + z^2) = \\ &(s''^2 + t''^2 + u''^2 + v''^2 + w''^2 + x''^2 + y''^2 + z''^2),\end{aligned}$$

or, as the theorem may be more concisely expressed,

$$\Sigma_8(\square) \times \Sigma_8(\square') = \Sigma_8(\square'').$$

Although the complete verification of this theorem has been actually exhibited as above, yet, as before briefly noticed, such verification was not absolutely necessary in order to produce confidence in the truth of the proposition. We have only to contemplate the internal constitution of the eight rows in the eight-square form here presented, in order to perceive that the conclusion just arrived at is a necessary consequence of that constitution. For, upon examining that form more minutely, we find that not only do the *consecutive* pairs of binomials in every row conform to the four-square model,—a degree of conformity which, as before observed, it is essential to secure,—but further, that in each row *every* pair of binomials, whether consecutive or not, involves two sets of four letters, such that if all the other letters in the entire form be suppressed, or be replaced by so many zeros, the model for four squares will be complied with by the remaining expressions. Now, as in every set of such remaining expressions the double products, furnished by development as above, would, as we know from the property of the four-square formula, cancel one another, it necessarily follows that the double products supplied by each of the eight rows of the preceding form must be cancelled by like double products arising from the other rows. It is solely because of this uniform agreement with the four-square model, when the quantities which compose a pair of binomials in any row are throughout supposed to be zero, that the eight-square form is admissible; for it is obvious that, if by means of these zeros any reduction of the advanced to the inferior form presented results not in conformity with the model for that inferior form, the double products furnished by such results could not cancel.

When once we are satisfied as to the existence of the eight-square form, we need not keep the above considerations constantly before us in the actual construction of it. After the first four rows are determined, the four that are to follow may be readily derived from them, by aid of the property that the like products are to cancel. Thus: having resolved to keep the leading signs plus, we commence the four rows, to be deduced from those above, with sw', sx', sy', sz', respectively; and, guided by the before-mentioned property, we annex to these, with the minus sign, the combinations ws', xs', ys', zs'. To determine the sign of xt', we refer to the combinations st', xw', in the second row; and as these give a *minus* product, we write down xt' *plus;* so that the product of sw', xt', in the row now in course of formation, may cancel the like product above: and by proceeding in this manner, all the wanting rows may be easily completed, each term that we insert suggesting the adjacent term. We thus see, when any formula of this kind actually exists, how the lower half of it may be deduced from the upper, without any further recurrence to a pre-established model. This is a circumstance worthy of note, as well as the following particulars respecting the several rows:

1. The same letter is never repeated in the same row.

2. The same combination of two letters is never repeated throughout the group of rows; and, consequently, the same product of two combinations can occur only twice.

3. In any row, the signs of any consecutive combinations, separated by the sign minus, may be interchanged without causing any departure from the preceding model: for such interchange of signs is merely equivalent to an interchange of places between the two dashed quantities entering the pair, and between the two undashed quantities; or it may be regarded as arising from making a letter in each of the two combinations negative, as we are, of course, at liberty to do, since the signs of the roots of the proposed squares,—and it is with these roots we are dealing,—are arbitrary. It must be observed, however, that these changes or interchanges, when made at all, must be made in every place where the letters concerned occur, otherwise conformity to the model will be destroyed.

4. It is thus obvious that the four-square and eight-square models admit of a great variety of apparently different forms. In each case, however, it is plain

that any one of these varieties, by suitable changes, such as those we have now mentioned, will supply any other; so that a single variety may be considered as virtually involving all the varieties: when the signs of the combinations are once so arranged as to cause the double products in the developments above to cancel, we may clearly change these signs in any way we please that does not interfere with this essential condition. Even after the *partial* construction of a formula, by aid of a previous model, as noticed at page 315, we may obviously introduce any changes of this kind that we please, provided we take care that all the like products, supplied by the partial form, shall still neutralize one another, the changes adverted to being, of course, made in obedience to the precept (3). We may then proceed to complete the formula in the way already explained.

Keeping these general principles in view, let us now reflect upon the construction of the sixteen-square formula. If such a construction be possible, we shall necessarily arrive at it by operating with sets of eight, under the guidance of the eight-square model, in imitation of the proceeding that conducted us to this latter formula from that for four. There can be no question that, if the supposed form exists, this is the legitimate and only sure path to it; inasmuch as we thus provide for the demands of the subordinate form while carrying on the construction of that next in advance,—a provision necessary to the correctness of the advanced form, since this must coincide with the former when reduced to it by the introduction of the proper number of zeros in place of the general symbols.

In this way, then, we shall obtain sixteen rows of products. These products will all differ from one another; and, as in the preceding cases, will be those which arise from combining all the proposed letters,—a dashed one with an undashed one,—in every possible way. And, assuming these combinations to be connected together, as in the former cases, by the signs fitted to cause all the double products to cancel, we may apply to the completed form, thus imagined, the remarks (1), (2), (3), (4), already made with more especial reference to the eight-square formula.

It is in accordance with these general principles and directions, as far, at least, as they can be complied with, that the following partial construction has been effected; and which, to the extent to which it has been carried, satisfies the

essential condition which it is our object to impress upon the completed form:— all the like products furnished by the combinations here exhibited cancel one another. If the anticipated form could be completed, then, as we have thus got more than half the requisite rows, such completion might now be effected simply by aid of these rows, and without any further reference to a guiding model, as already sufficiently explained at page 321. But this we shall find to be impossible; for although, as just stated, all the like products supplied by the partial construction really cancel one another, yet there is inherent in that construction a want of conformity with the eight-square model, as respects one of the sets of eight involved, which no modifications of signs can remove; and which, by overruling the combinations yet to be formed, precludes the cancelling of the like products which remain to be compared. This will be clearly seen presently.

Partial Construction of the sixteen-square Formula.

$$s'+t'+u'+v'+w'+x'+y'+z'+s'_{,}+t'_{,}+u'_{,}+v'_{,}+w'_{,}+x'_{,}+y'_{,}+z'_{,}$$
$$s+t+u+v+w+x+y+z+s_{,}+t_{,}+u_{,}+v_{,}+w_{,}+x_{,}+y_{,}+z'$$

$$\begin{array}{l}
ss'+tt'+uu'+vv'+ww'+xx'+yy'+zz'+s_{,}s'_{,}+t_{,}t'_{,}+u_{,}u'_{,}+v_{,}v'_{,}+w_{,}w'_{,}+x_{,}x'_{,}+y_{,}y'_{,}+z_{,}z'_{,}\\
st'-ts'+uv'-vu'+wx'-xw'+yz'-zy'+s_{,}t'_{,}-t_{,}s'_{,}+u_{,}v'_{,}-v_{,}u'_{,}+w_{,}x'_{,}-x_{,}w'_{,}+y_{,}z'_{,}-z_{,}y'_{,}\\
su'-us'+vt'-tv'+yw'-wy'+xz'-zx'+s_{,}u'_{,}-u_{,}s'_{,}+v_{,}t'_{,}-t_{,}v'_{,}+w_{,}y'_{,}-y_{,}w'_{,}+z_{,}x'_{,}-x_{,}z'_{,}\\
sv'-vs'+tu'-ut'+wz'-zw'+xy'-yx'+v_{,}s'_{,}-s_{,}v'_{,}+u_{,}t'_{,}-t_{,}u'_{,}+w_{,}z'_{,}-z_{,}w'_{,}+x_{,}y'_{,}-y_{,}x'_{,}\\
sw'-ws'+xt'-tx'+uy'-yu'+zv'-vz'+s_{,}w'_{,}-w_{,}s'_{,}+x_{,}t'_{,}-t_{,}x'_{,}+y_{,}u'_{,}-u_{,}y'_{,}+v_{,}z'_{,}-z_{,}v'_{,}\\
sx'-xs'+tw'-wt'+yv'-vy'+zu'-uz'+s_{,}x'_{,}-x_{,}s'_{,}+t_{,}w'_{,}-w_{,}t'_{,}+v_{,}y'_{,}-y_{,}v'_{,}+u_{,}z'_{,}-z_{,}u'_{,}\\
sy'-ys'+zt'-tz'+vx'-xv'+wu'-uw'+y_{,}s'_{,}-s_{,}y'_{,}+t_{,}z'_{,}-z_{,}t'_{,}+v_{,}x'_{,}-x_{,}v'_{,}+w_{,}u'_{,}-u_{,}w'_{,}\\
sz'-zs'+ty'-yt'+vw'-wv'+ux'-xu'+s_{,}z'_{,}-z_{,}s'_{,}+t_{,}y'_{,}-y_{,}t'_{,}+w_{,}v'_{,}-v_{,}w'_{,}+x_{,}u'_{,}-u_{,}x'_{,}\\
ss'_{,}-s_{,}s'+t_{,}t'-tt'_{,}+u_{,}u'-uu'_{,}+vv'_{,}-v_{,}v'\\
st'_{,}-t_{,}s'+ts'_{,}-s_{,}t'+vu'_{,}-u_{,}v'+uv'_{,}-v_{,}u'\\
su'_{,}-u_{,}s'+v_{,}t'-tv'_{,}+t_{,}v'-vt'_{,}+us'_{,}-s_{,}u'\\
sv'_{,}-v_{,}s'+tu'_{,}-u_{,}t'+s_{,}v'-vs'_{,}+t_{,}u'-ut'_{,}.^{*}
\end{array}$$

Now, it appears from examining these expressions, that, as before remarked,

* These rows are exhibited complete in the scholium at the end of this paper, where it is shown that, under certain conditions, the sixteen-square formula has place.

the object in view is, to a certain extent, accomplished, for all the like products cancel one another. If to this we could add that, as far as they go, these same expressions agree with the eight-square model, then we might at once conclude the existence of the sixteen-square formula, and might actually complete it by deducing the wanting rows from those above, without disturbing any of the signs of these latter, as already explained at page 321.

But the semi-row next in order to the last of the above group, as deduced in this way from the preceding expressions, is

$$sw'_{,} - w_{,}s' + x_{,}t' - tx'_{,} + y_{,}u' - uy'_{,} + z_{,}v' - vz'_{,},$$

which implies a discrepancy; for the product of the combinations $tx'_{,}$, $zv'_{,}$, does not cancel the like product furnished by tv', $z_{,}x'_{,}$, in the third row, the signs of both products being *minus*. We are forced to conclude, therefore, that, notwithstanding the cancelling of all the like products in the partial construction above, there is, at least, one set of eight which, in that construction, is out of keeping with the model; and it now remains for us to discover this set, and to inquire whether it can possibly be brought into conformity with the model, without such an interference with the existing signs as would cause products which already vanish to re-appear.

If we refer to our eight-square model, and conceive the semi-rows,

$$w' + x' + y' + z'$$
$$w + x + y + z,$$

there employed, to be changed into

$$w'_{,} + x'_{,} + y'_{,} + z'_{,}$$
$$w_{,} + x_{,} + y_{,} + z_{,},$$

the modified formula will then agree,—as far as the first four rows, which are all that are here exhibited,—with the expressions which, in the above scheme, make up the commencing and terminating portions of the first four rows, with this important difference, namely, that in the terminating portion of the third row the signs are the opposites of those required by the model. This, as we shall presently see, is the discrepancy already indicated; and we proceed to show that its character is such that it cannot possibly be removed without the introduction of a similar discrepancy elsewhere; that is to say, the refractory signs here alluded to cannot be brought into conformity with the law of the model,—as they must

be before the products furnished by the particular set of eight now considered can cancel,—without introducing discord among other sets of eight, at present in harmony with that model, and thus causing products to re-appear which now cancel.

In order to this, we may observe that there are only two different ways of proceeding, by which the required modification of the signs of

$$w_, y'_, - y_, w'_, + z_, x'_, - x_, z'_,$$

can be brought about; first, by supposing two pairs of letters in the expression to be minus, as $w_,$, $w'_,$, and $x_,$, $x'_,$, or $z_,$, $z'_,$; or, instead of $w_,$, $w'_,$, taking y', $y'_,$, with either of the latter pairs; and, secondly, by changing all the signs in the semi-row,

$$s_, u'_, - u_, s'_, + v_, t'_, - t_, v'_, + w_, y'_, - y_, w'_, + z_, x'_, - x_, z'_,.$$

Suppose we attempt to remove the discrepancy in the former of these ways, that is, by simply changing the signs of $w_,$, $w'_,$, and $x_,$, $x'_,$, or $z_,$, $z'_,$.

If we make no further changes, conformity to the model, for the before-mentioned set of eight, will unquestionably be brought about; but in order to preserve the conformity which already exists for the set,

$$s'_, + t'_, + u'_, + v'_, + w'_, + x'_, + y'_, + z'_,$$
$$s_, + t_, + u_, + v_, + w_, + x_, + y_, + z_,$$

these same changes of signs must also be made in the rows which precede and follow the third in the above scheme: but these changes in the other rows introduce the discrepancy into the first-mentioned set, or rather merely transfer it to the neighbouring row in that set. Of course, changing the signs of any quantities, *throughout* any of these formulæ, can never interfere with any conformity or want of conformity to a model that might exist before these changes were made. Again, if we attempt the correction by changing all the signs in the semi-row,

$$s_, u'_, - u_, s'_, + v_, t'_, - t_, v'_, + w_, y'_, - y_, w'_, + z_, x'_, - x_, z'_,,$$

leaving the other signs unaltered, the desired conformity will also in this way be brought about; and we shall, at the same time, escape the dilemma in which the former mode of proceeding placed us; for the second-mentioned set of eight will still preserve its conformity to the model, inasmuch as the changing of the signs of an entire row can never disturb such conformity. But here again disagree-

ment is introduced in another place, for the semi-row commencing with $ss'_{,}$ would then be

$$ss'_{,} - s_{,}s' + t_{,}t' - tt'_{,} + uu'_{,} - u_{,}u' + vv'_{,} - v_{,}v',$$

so that the product of the combinations $tt'_{,}$, $u_{,}u'$, would no longer cancel the product of tu', $u_{,}t'_{,}$, furnished by the fourth row; and thus, as before, the removal of one discrepancy necessitates the introduction of another.

It is this discrepancy, as respects the last four signs in the third row above, that causes the signs in the semi-row mentioned at page 324, viz.:

$$sw'_{,} - w_{,}s' + x_{,}t' - tx'_{,} + y_{,}u' - uy'_{,} + z_{,}v' - vz'_{,},$$

and which were in part deduced from them, to involve error; an error which, as we have seen, cannot possibly be removed without introducing a like error elsewhere.

Hence, the sixteen-square formula is, in general, impossible; and from this it follows that the thirty-two square form is also impossible, and so on. For if the form for thirty-two had place, then, by reducing these thirty-two to sixteen, by employing zeros instead of the remaining quantities, we should be conducted to a correct sixteen-square formula, which has been shown, however, to have no existence.

If it be imagined for a moment that an advanced form might have place, and yet the next inferior form not be necessarily furnished by it when the requisite squares are assumed to be zero, the impression will be removed by observing that when, in the form for four, the two dashed and the two undashed quantities entering into any of the binomials are throughout made zero, two entire rows of that form disappear; that when, in the form for eight, the four dashed and the four undashed quantities entering into a pair of binomials in any row, are throughout made zero, four entire rows vanish; and, likewise, in the failing form for sixteen, and generally, the necessary constitution of the rows renders these evanescences unavoidable.

It is proper that we make the preceding stipulation as to the zero-quantities forming binomials in the same row; for if they be chosen at random, the above conclusion will not necessarily follow. In the eight-square formula, for instance, if our zero-quantities are not selected in reference to this condition, only one row will disappear: thus, if the quantities made zero be s, t, x, y,

and s', t', x', y', the row marked (u'') will be the only row that will vanish. We may hence notice, in passing, that four squares multiplied by four squares may be made to produce *seven* squares; but not, in general, either six squares or five.

We shall only further remind the reader that, when any formula of the kind here discussed actually exists, and that we have partially constructed it, to an extent however limited, in strict conformity to the subordinate model, that construction must be correct as far as it goes; and must admit of completion without disturbing the signs in the partial form. Whenever, therefore, such completion is shown to be impossible, we may infer that the supposed form has no existence; and from this consideration, by first excluding the four combinations here shown to be refractory from the above group, we may arrive, somewhat differently, at the conclusion already deduced. We shall again advert to this presently.

Returning now to the four-square and eight-square formulæ, we may make the following inferences, viz.:

1. Certain coefficients may be introduced in connexion with the original squares, which coefficients will reappear in the corresponding squares of the product. This will readily be seen by taking the case of four squares with the suitable coefficients,

$$\begin{array}{c} w' + \sqrt{b}\,.\,x' + \sqrt{c}\,.\,y' + \sqrt{(bc)}\,.\,z' \\ w + \sqrt{b}\,.\,x + \sqrt{c}\,.\,y + \sqrt{(bc)}\,.\,z \\ \hline ww' + bxx' + cyy' + bczz' = w'' \\ \sqrt{b}(wx' - xw' + czy' - cyz') = \sqrt{b}\,.\,x'' \\ \sqrt{c}(wy' - yw' + bzx' - bxz') = \sqrt{c}\,.\,y'' \\ \sqrt{(bc)}(wz' - zw' + xy' - yx') = \sqrt{(bc)}\,.\,z'', \end{array}$$

which shows, as indeed was before proved by Lagrange, that the product of

$$(w'^2 + bx'^2 + cy'^2 + bcz'^2)$$

and

$$(w^2 + bx^2 + cy^2 + bcz^2)$$

is of the same form as each of the factors; that is

$$(w'^2 + bx'^2 + cy'^2 + bcz'^2)\,(w^2 + bx^2 + cy^2 + bcz^2) = w''^2 + bx''^2 + cy''^2 + bcz''^2.$$

And from the construction of the eight-square formula, it is further evident that the product

$$\begin{aligned}&(s'^2 + bt'^2 + cu'^2 + bcv'^2 + bcw'^2 + cx'^2 + by'^2 + z'^2) \times\\ &(s^2 + bt^2 + cu^2 + bcv^2 + bcw^2 + cx^2 + by^2 + z^2) =\\ &(s''^2 + bt''^2 + cu''^2 + bcv''^2 + bcw''^2 + cx''^2 + by''^2 + z''^2);\end{aligned}$$

that is, it is of the same form as the original factors.

2. If $c = b^2$, then each of these factors is of the form

$$s^2 + bt^2 + b^2u^2 + b^3v^2 + b^3w^2 + b^2x^2 + by^2 + z^2.$$

But of this same form also is

$$s_{,}^2 + at_{,}^2 + a^2u_{,}^2 + a^3v_{,}^2 + a^4w_{,}^2 + a^5x_{,}^2 + a^6y_{,}^2 + a^7z_{,}^2,$$

inasmuch as this is convertible into

$$s_{,}^2 + at_{,}^2 + a^2u_{,}^2 + a^3v_{,}^2 + a^3(a^2z')^2 + a^2(a^2y_{,})^2 + a(a^2x_{,})^2 + (a^2w_{,})^2.$$

Hence the product continues to be of the same form as each of the factors, when the coefficients a^0, a^1, a^2, a^3, &c., are introduced in order, in connexion with the squares entering those factors. And it is plain that a may be either positive or negative, real or imaginary.

3. If a be equal to -1, the factors will consist of squares alternately positive and negative: therefore, the squares in the product will also be alternately positive and negative. Consequently, the difference of two squares, multiplied by the difference of two squares, will produce the difference of two squares: the sum of two squares minus the sum of two squares, multiplied by the sum of two squares minus the sum of two squares, will produce the sum of two squares minus the sum of two squares: and the sum of four squares minus the sum of four squares, will produce the sum of four squares minus the sum of four squares.

4. If in the forms for four and eight we make a pair, or any number of pairs, of *like* letters negative, it is plain that the first row of results will remain unchanged, while the other rows (unless the changes of signs adverted to extend to *all* the letters) will undergo alterations; yet, from the modular property still having place, it follows that the sum of the squares of these other rows will continue invariable, and thus we shall get different forms for the sum of *three* squares and for the sum of *seven* squares; these sums being all equivalent, although the

individual squares differ. And similar results have place, when the coefficients hitherto used in connexion with these individual squares are introduced. In like manner, if we make any *one* or more of the letters in the original factors negative, the form we shall thus get will be such, that the sum of the squares of the rows, omitting the first row, will always be identical to the sum of the squares of the same rows, when pairs of like letters—any whatever—are made negative as before.

5. If we refer to the four-square form, and, omitting the first row of results, multiply the binomials in each of the three remaining rows together, the sum of the three products thus obtained will always be zero, whatever variety of the general form we use. This will appear from observing that the monomial products furnished by these binomial factors, are all unlike those furnished by the square of the omitted first row; so that they must be such as to cancel one another, and thus cause the entire aggregate of the results to be zero.

6. Similar conclusions apply to the eight-square form: thus, if we represent the four binomials in the second row of that form by t_1'', t_2'', t_3'', t_4''; those in the next row by u_1'', u_2'', u_3'', u_4''; and so on to the last row, z_1'', z_2'', z_3'', z_4''; we shall have the property,

$$\begin{aligned} & t_1''(t_2'' + t_3'' + t_4'') + t_2''(t_3'' + t_4'') + t_3''t_4'' \\ + & u_1''(u_2'' + u_3'' + u_4'') + u_2''(u_3'' + u_4'') + u_3''u_4'' \\ & \cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots \\ + & z_1''(z_2'' + z_3'' + z_4'') + z_2''(z_3'' + z_4'') + z_3''z_4'' = 0. \end{aligned}$$

And if we represent the several rows here written by the single symbols T, U, $\mathrm{V}, \ldots \mathrm{Z}$, and regard b and c to be involved in them, as already shown in the general form, it is equally true that

$$b\mathrm{T} + c\mathrm{U} + bc\mathrm{V} + bc\mathrm{W} + c\mathrm{X} + b\mathrm{Y} + \mathrm{Z} = 0.$$

7. Lastly, if we call the first row of results S, it further follows, from these latter inferences, that the product of the original squares will be represented by

$$\mathrm{S}^2 + t_1''^2 + t_2''^2 + t_3''^2 + t_4''^2 + \ldots + z_1''^2 + z_2''^2 + z_3''^2 + z_4''^2,$$

so that, omitting S^2, we see that the sum of the squares of all the rows, after the first, is equal to the sum of the squares of all the *binomials* which compose those

rows, as may also be inferred from the researches of Cauchy, hereafter mentioned.

It is proper to state, that the inferences marked (5), (6), (7), are analogous to, and were suggested by those of Sir William R. Hamilton, at pages 60 and 68* of his "Researches respecting Quaternions," in the present Part of the Transactions; with a copy of which Researches I was favoured while the communication now before the reader (the above-mentioned articles excepted) was in the hands of the Academy.

8. To these inferences we may add that three squares, multiplied by three squares, will produce three squares, provided a square in one factor have to a square in the other the same ratio that a second square in the former has to a second in the latter; for, in this case, as it is easy to see, an entire row of combinations will disappear from the four-square form.† Similarly for seven squares, if three such equal ratios occur. And we may readily ascertain the corresponding conditions for six and five. But the consequences of particular hypotheses of this kind will be more fully noticed in the following supplementary observations.

* Pages 258 and 266 of Vol. XXI., Part 2.

† The sum of three squares, multiplied by the sum of three squares, will also produce the sum of three squares, provided the factors be so related that the first row of results in the four-square construction vanish. This relation is furnished by the co-ordinates of the extremities of a system of semi-conjugates in an ellipsoid.

For, denoting these extremities by (x', y', z'), (x'', y'', z''), (x''', y''', z'''), it is a known property of the surface that

$$(y'^2 + y''^2 + y'''^2)(x'^2 + x''^2 + x'''^2) = (x'y'' - x''y')^2 + (x'y''' - x'''y')^2 + (x''y''' - x'''y'')^2;$$

and, consequently, from our four-square form, we must have

$$x'y' + x''y'' + x'''y''' = 0,$$

and similarly

$$x'z' + x''z'' + x'''z''' = 0,$$

and

$$y'z' + y''z'' + y'''z''' = 0;$$

and thus is suggested a neat way of deriving several properties of the surface. But geometrical application is not our object in the present paper.

[In the proof of this sheet I think it right to add, that these three equations are otherwise obtained by Mr. Weddle, in a paper on the Ellipsoid, published in the Cambridge Mathematical Journal for January, 1847, but not seen by me till my copy was in the hands of the printer.]

SCHOLIA.

1. Throughout the general reasonings in the foregoing discussion, we have considered the squares which enter one of the proposed factors to be different in value from those which enter the other. When, however, this is not the case, and the two factors are identical, it may be proper here to observe that, in using any of the preceding formulæ for the product, one, at least, of the root-quantities furnished by the multiplier, or one of those furnished by the multiplicand, must be taken with the minus sign; otherwise, in the product, only the first row of the results will be significant; and thus no decomposition of the proposed square will be obtained. But there is no need for any model formula in this case; for it is pretty obvious that the square of a polynomial, formed by the sum of any number of squares, may be expressed by a polynomial of the same number of squares, without any limitation: thus,

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2 + \dots + x_n^2)^2 =$$
$$(x_1^2 - x_2^2 + x_3^2 + x_4^2 + \dots + x_n^2)^2 + (2x_2x_1)^2 + (2x_2x_3)^2 + (2x_2x_4)^2 + \dots + (2x_2x_n)^2.$$

The same thing evidently holds when the factors, instead of being identical, are such, that the several squares in the one have the same common ratio to those in the other. And formulæ might be determined, of like generality with this, which would exhibit the product when only a partial number of these ratios are equal.

For instance, the product of sixteen squares by sixteen, will be expressed by the sum of the squares of the sixteen rows of combinations which follow; provided there exist these eight equal ratios, viz.:

$$\frac{s_{\prime}}{s'_{\prime}} = \frac{t_{\prime}}{t'_{\prime}} = \frac{u_{\prime}}{u'_{\prime}} = \frac{v_{\prime}}{v'_{\prime}} = \frac{w_{\prime}}{w'_{\prime}} = \frac{x_{\prime}}{x'_{\prime}} = \frac{y_{\prime}}{y'_{\prime}} = \frac{z_{\prime}}{z'_{\prime}} \dots (1),$$

among the proposed quantities,

$$\left.\begin{array}{l} (s'^2+t'^2+u'^2+v'^2+w'^2+x'^2+y'^2+z'^2+s'^2_{\prime}+t'^2_{\prime}+u'^2_{\prime}+v'^2_{\prime}+w'^2_{\prime}+x'^2_{\prime}+y'^2_{\prime}+z'^2_{\prime}) \\ (s^2+t^2+u^2+v^2+w^2+x^2+y^2+z^2+s^2_{\prime}+t^2_{\prime}+u^2_{\prime}+v^2_{\prime}+w^2_{\prime}+x^2_{\prime}+y^2_{\prime}+z^2_{\prime}) \end{array}\right\} \dots (2).$$

Expressions, of which the Sum of the Squares is equal to the Product of the Factors (2), *when the Conditions* (1) *have place.*

$$
\begin{array}{l}
ss'+tt'+uu'+vv'+ww'+xx'+yy'+zz'+s_{\prime}s'_{\prime}+t_{\prime}t'_{\prime}+u_{\prime}u'_{\prime}+v_{\prime}v'_{\prime}+w_{\prime}w'_{\prime}+x_{\prime}x'_{\prime}+y_{\prime}y'_{\prime}+z_{\prime}z'_{\prime},\\
st'-ts'+uv'-vu'+wx'-xw'+yz'-zy'+s_{\prime}t'_{\prime}-t_{\prime}s'_{\prime}+u_{\prime}v'_{\prime}-v_{\prime}u'_{\prime}+w_{\prime}x'_{\prime}-x_{\prime}w'_{\prime}+y_{\prime}z'_{\prime}-z_{\prime}y'_{\prime},\\
su'-us'+vt'-tv'+yw'-wy'+xz'-zx'+s_{\prime}u'_{\prime}-u_{\prime}s'_{\prime}+v_{\prime}t'_{\prime}-t_{\prime}v'_{\prime}+w_{\prime}y'_{\prime}-y_{\prime}w'_{\prime}+z_{\prime}x'_{\prime}-x_{\prime}z'_{\prime},\\
sv'-vs'+tu'-ut'+wz'-zw'+xy'-yx'+v_{\prime}s'_{\prime}-s_{\prime}v'_{\prime}+u_{\prime}t'_{\prime}-t_{\prime}u'_{\prime}+w_{\prime}z'_{\prime}-z_{\prime}w'_{\prime}+x_{\prime}y'_{\prime}-y_{\prime}x'_{\prime},\\
sw'-ws'+xt'-tx'+uy'-yu'+zv'-vz'+s_{\prime}w'_{\prime}-w_{\prime}s'_{\prime}+x_{\prime}t'_{\prime}-t_{\prime}x'_{\prime}+y_{\prime}u'_{\prime}-u_{\prime}y'_{\prime}+v_{\prime}z'_{\prime}-z_{\prime}v'_{\prime},\\
sx'-xs'+tw'-wt'+yv'-vy'+zu'-uz'+s_{\prime}x'_{\prime}-x_{\prime}s'_{\prime}+t_{\prime}w'_{\prime}-w_{\prime}t'_{\prime}+v_{\prime}y'_{\prime}-y_{\prime}v'_{\prime}+u_{\prime}z'_{\prime}-z_{\prime}u'_{\prime},\\
sy'-ys'+zt'-tz'+vx'-xv'+wu'-uw'+y_{\prime}s'_{\prime}-s_{\prime}y'_{\prime}+t_{\prime}z'_{\prime}-z_{\prime}t'_{\prime}+v_{\prime}x'_{\prime}-x_{\prime}v'_{\prime}+w_{\prime}u'_{\prime}-u_{\prime}w'_{\prime},\\
sz'-zs'+ty'-yt'+vw'-wv'+ux'-xu'+s_{\prime}z'_{\prime}-z_{\prime}s'_{\prime}+t_{\prime}y'_{\prime}-y_{\prime}t'_{\prime}+w_{\prime}v'_{\prime}-v_{\prime}w'_{\prime}+x_{\prime}u'_{\prime}-u_{\prime}x'_{\prime},\\
ss'_{\prime}-s_{\prime}s'+t_{\prime}t'-tt'_{\prime}+u_{\prime}u'-uu'_{\prime}+vv'_{\prime}-v_{\prime}v'+w'w_{\prime}-ww'_{\prime}+x_{\prime}x'-xx'_{\prime}+yy'_{\prime}-y_{\prime}y'+z_{\prime}z'-zz'_{\prime},\\
st'_{\prime}-t_{\prime}s'+ts'_{\prime}-s_{\prime}t'+vu'_{\prime}-u_{\prime}v'+uv'_{\prime}-v_{\prime}u'+wx'_{\prime}-x_{\prime}w'+w_{\prime}x'-xw'_{\prime}+y_{\prime}z'-zy'_{\prime}+z_{\prime}y'-yz'_{\prime},\\
su'_{\prime}-u_{\prime}s'+v_{\prime}t'-tv'_{\prime}+t_{\prime}v'-vt'_{\prime}+us'_{\prime}-s_{\prime}u'+wy'_{\prime}-y_{\prime}w'+z_{\prime}x'-xz'_{\prime}+zx'_{\prime}-x_{\prime}z'+yw'_{\prime}-w_{\prime}y'\\
sv'_{\prime}-v_{\prime}s'+tu'_{\prime}-u_{\prime}t'+s_{\prime}v'-vs'_{\prime}+t_{\prime}u'-ut'_{\prime}+x_{\prime}y'-yx'_{\prime}+z_{\prime}w'-wz'_{\prime}+y_{\prime}x'-xy'_{\prime}+zw'_{\prime}-w_{\prime}z'\\
sw'_{\prime}-w_{\prime}s'+x_{\prime}t'-tx'_{\prime}+y_{\prime}u'-uy'_{\prime}+z_{\prime}v'-vz'_{\prime}+ws'_{\prime}-s_{\prime}w'+xt'_{\prime}-t_{\prime}x'+u_{\prime}y'-yu'_{\prime}+v_{\prime}z'-zv'_{\prime},\\
sx'_{\prime}-x_{\prime}s'+tw'_{\prime}-w_{\prime}t'+y_{\prime}v'-vy'_{\prime}+uz'_{\prime}-z_{\prime}u'+t_{\prime}w'-wt'_{\prime}+xs'_{\prime}-s_{\prime}x'+u_{\prime}z'-zu'_{\prime}+yv'_{\prime}-v_{\prime}y'\\
sy'_{\prime}-y_{\prime}s'+z_{\prime}t'-tz'_{\prime}+vx'_{\prime}-x_{\prime}v'+uw'_{\prime}-w_{\prime}u'+xv'_{\prime}-v_{\prime}x'+u_{\prime}w'-wu'_{\prime}+zt'_{\prime}-t_{\prime}z'+s_{\prime}y'-ys'_{\prime},\\
sz'_{\prime}-z_{\prime}s'+ty'_{\prime}-y_{\prime}t'+vw'_{\prime}-w_{\prime}v'+x_{\prime}u'-ux'_{\prime}+wv'_{\prime}-v_{\prime}w'+xu'_{\prime}-u_{\prime}x'+zs'_{\prime}-s_{\prime}z'+yt'_{\prime}-t_{\prime}y'
\end{array}
$$

Since, in virtue of the conditions (1), the semi-row commencing with $s_{\prime}t'_{\prime}$, and the six semi-rows which follow it, all vanish, they should be expunged from the above form, after they have subserved the purpose of facilitating the construction of the lower rows, agreeably to the directions at page 321.

Dismissing, then, these zero-binomials as superfluous, we shall find all the double products, arising from the development of the squares of the sixteen rows of combinations, to disappear, like as in the forms before established. Those products whose like occur in the group here supposed to be expunged, will be cancelled by equal products in the group retained: these latter products, though not *like*, being *equivalent* in virtue of the assumed conditions (1) above. And it may be further observed that, after the suppression here recommended, the present form may be employed, instead of the more abridged one at page 323, to prove the general impossibility of the sixteen-square theorem; for it may be readily shown, in a manner analogous to that there adopted, that the chasm

thus left cannot be filled up so as to satisfy the conditions necessary to the existence of the theorem in its general form.

2. If we glance at the developments exhibited at length for the eight-square form, or contemplate the leading binomials in the several rows of that above, we shall immediately perceive that the squares of these leading binomials furnish all the double products which cancel those supplied by the square of the first row of combinations; and that such must always be the case whatever number of squares enter the original factors. We infer, therefore, that

$$(x_1^2+x_2^2+x_3^2+\ldots\ldots+x_n^2)\,(y_1^2+y_2^2+y_3^2+\ldots\ldots y_n^2)=$$
$$(x_1y_1+x_2y_2+x_3y_3+\ldots\ldots+x_ny_n)^2+(x_1y_2-x_2y_1)^2+(x_1y_3-x_3y_1)^2+\ldots\ldots+(x_1y_n-x_ny_1)^2+\mathrm{P}_1,$$

where P_1 is some function of the combinations, into which, however, neither x_1 nor y_1 can enter, or, at least, can enter only to be mutually neutralized, since all the combinations involving these are evidently implied in the other terms.

Suppose, now, that $x_1=0$, and $y_1=0$; then, similarly,

$$(x_2^2+x_3^2+x_4^2+\ldots\ldots+x_n^2)\,(y_2^2+y_3^2+y_4^2+\ldots\ldots+y_n^2)=$$
$$(x_2y_2+x_3y_3+x_4y_4+\ldots\ldots+x_ny_n)^2+(x_2y_3-x_3y_2)^2+(x_2y_4-x_4y_2)^2+\ldots\ldots+(x_2y_n-x_ny_2)^2+\mathrm{P}_2.$$

But, on the same supposition, the preceding equation gives, for the first member of this, the value

$$(x_2y_2+x_3y_3+x_4y_4+\ldots\ldots+x_ny_n)^2+\mathrm{P}_1,$$

since P_1 is not affected by the supposition. Hence

$$\mathrm{P}_1=(x_2y_3-x_3y_2)^2+(x_2y_4-x_4y_2)^2+\ldots\ldots+(x_2y_n-x_ny_2)^2+\mathrm{P}_2;$$

Similarly,

$$\mathrm{P}_2=(x_3y_4-x_4y_3)^2+(x_3y_5-x_5y_3)^2+\ldots\ldots+(x_3y_n-x_ny_3)^2+\mathrm{P}_3,$$

and so on. And thus, by supposing successively $x_2=0$, $y_2=0$; $x_3=0$, $y_3=0$; &c., up to $x_n=0$, $y_n=0$; we shall arrive at $\mathrm{P}_n=0$; so that, returning, by successive substitutions, to the original equation, we have finally

$$\begin{aligned}&(x_1^2+x_2^2+x_3^2+\ldots\ldots+x_n^2)\,(y_1^2+y_2^2+y_3^2+\ldots\ldots+y_n^2)=\\&\qquad(x_1y_1+x_2y_2+x_3y_3+\ldots\ldots+x_ny_n)^2+\\&(x_1y_2-x_2y_1)^2+(x_1y_3-x_3y_1)^2+\ldots\ldots+(x_1y_n-x_ny_1)^2+\\&(x_2y_3-x_3y_2)^2+(x_2y_4-x_4y_2)^2+\ldots\ldots+(x_2y_n-x_ny_2)^2+\\&(x_3y_4-x_4y_3)^2+(x_3y_5-x_5y_3)^2+\ldots\ldots+(x_3y_n-x_ny_3)^2+\ldots\ldots(x_{n-1}y_n-x_ny_{n-1})^2,\end{aligned}$$

a formula which has been otherwise established by Cauchy, in his "Cours d'Analyse," page 455. Cauchy adds, that when $\frac{x_1}{y_1}, \frac{x_2}{y_2}$, &c., are equal, the second member is reduced to its first term. But this is only saying that $f(mx)^2 = f(m^2x^2)$. The foregoing equation warrants the inference (7), as already noticed.

3. Before concluding these researches, the author is desirous of adding a word or two on Sir William R. Hamilton's Quaternions; a subject which, as already remarked at the outset, is intimately connected with some of the speculations in the present paper.

If we refer to the four rows of combinations to which we have been conducted by the forms $(w' + \sqrt{b} \,.\, x' + \sqrt{c} \,.\, y' + \sqrt{(bc)} \,.\, z')\,(w + \sqrt{b} \,.\, x + \sqrt{c} \,.\, y + \sqrt{(bc)} \,.\, z)$, at page 327, we shall perceive that, with the exception of the signs, these combinations make up the actual product which would arise from multiplying those forms together as factors. This circumstance is calculated to suggest the inquiry, whether, by imposing certain laws of combination, in reference to the coefficients $\sqrt{b}$, $\sqrt{c}$, $\sqrt{(bc)}$, the rows alluded to might not be made to represent the product, signs and all. Such an inquiry would lead us to the conditions originally proposed by Sir William Hamilton; for, changing $\sqrt{b}$, $\sqrt{c}$, $\sqrt{(bc)}$, into i, j, k, we should find that the laws of combination to which these symbols must be subject, to produce the desired effect, are those implied in the following relations, viz.:

$$i^2 = j^2 = k^2 = -1\,;$$
$$ij = k\,;\ jk = i\,;\ ki = j\,;$$
$$ji = -k\,;\ kj = -i\,;\ ik = -j.$$

With these symbols, under these relations, the factors referred to are *quaternions;* and we see that the product of two quaternions must produce a quaternion. And this is the fundamental theorem from which Sir William Hamilton has deduced so many and such remarkable results.

If we were to content ourselves with a very narrow and imperfect view of the office of the above symbols, we might regard them simply as ingenious contrivances for facilitating the construction of the four-square formula; since, after the combinations which make up the product of two quaternions have been obtained, conformably to the foregoing relations, we may then change our temporary symbols back again to their originals, $\sqrt{b}$, $\sqrt{c}$, $\sqrt{(bc)}$, and thus easily

recover the four-square form, or some variety of that form, should it have escaped the memory. In this limited point of view, quaternions would bear some analogy, as respects the office performed, to the well-known contrivance of Napier for the solution of right-angled spherical triangles.

But such a temporary and isolated purpose as this is not one to which quaternions are confined in Sir William Hamilton's more comprehensive theory of these expressions, in which theory they are extensively employed as new and permanent instruments of analysis. The laws to which they are subject are, as we perceive from the foregoing fundamental and essential relations, altogether different from those which govern the operations of common algebra; and, therefore, viewed as a part of common algebra, they could not be admitted. But it must be remembered that the symbols i, j, k, introduced into this theory, are confessedly distinct in meaning from those recognised in the older algebra; and are thus governed by laws peculiar to themselves. Everything in the previously existing algebra is left undisturbed; there is no *innovation*, nor, strictly speaking, any *extension* of its hitherto admitted principles: it is rather the *addition* to it of an algebra altogether new. Descartes, by giving a new office to certain marks employed in algebra, acquired additional power over symbolical geometry: with him the signs $+$ and $-$ were used to denote *geometrical position*. In Sir William Hamilton's theory, these same signs serve to distinguish *algebraical* position, or order of succession; and, in a product, mark the difference between taking one of the factors for a multiplier, and the other,—a distinction for which the ordinary algebra does not provide, but which is found to suggest new paths of inquiry both in pure and applied science. Carnot opposed the Cartesian doctrine from an imperfect view of its peculiar character, and from overlooking its avowed stipulations, as the author of this paper has attempted to show elsewhere;* and it is possible that the theory of quaternions may be regarded with suspicion from like causes:—from an imagined discrepancy, namely, between its operations and the operations of the common algebra; it being altogether overlooked that the quantities peculiar to a quaternion are wholly distinct from those hitherto received into algebra, and, therefore, are not necessarily amenable to its

* Mathematical Dissertations, Diss. I.

laws: in fact, it is their non-subjection to these laws that alone constitutes their novelty. But the author of the present paper makes these remarks on the fundamental character of the quaternion symbols with diffidence, lest he should, in any degree, obscure what has been so often and so fully explained by the distinguished propounder of the quaternion theory himself. He will merely observe, in conclusion, that a form somewhat more general may be given to the original quaternion theorem; for we may write it thus, viz.:

$$(w + ibx + jcy + kbcz)(w' + ibx' + jcy' + kbcz') = \\ w'' + ibx'' + jcy'' + kbcz'',$$

as is evident from the construction at page 327, $\sqrt{b}$, $\sqrt{c}$, $\sqrt{(bc)}$, being changed into b, c, bc.

NOTE.—Mr. John T. Graves's demonstration of the eight-square theorem, and which, I believe, has not as yet been published, was, I learn, conducted by aid of the quaternion theory, modified by the introduction of certain additional imaginary elements:—I believe, four.* Without such additional symbols it would seem that the theorem could not be established in this way; as it is not the case that the sum of two quaternions, multiplied by the sum of two quaternions, produces the sum of two quaternions. It would be interesting to see the quaternion calculus extended to "octaves;" and it is to be hoped that Mr. Graves may be prevailed upon to make his researches on this subject public. They would probably suggest a generalization of the coefficients b, c, bc, introduced, in the foregoing paper, into the eight-square formula; just as the coefficients here adverted to, when employed in the form for fours, might have suggested the quaternions, as already noticed above.

The only more general form for these coefficients which occurs to myself, is

* See additional note at the end of this Paper.

that which arises from the introduction of another arbitrary factor into the last four terms of each octave; for such new factor may be introduced without any infringement of the modular condition. Thus,

$$\begin{aligned}&(s'^2 + bt'^2 + cu'^2 + bcv'^2 + abcw'^2 + acx'^2 + aby'^2 + az'\,) \times \\ &(s^2 + bt^2 + cu^2 + bcv^2 + abcw^2 + acx^2 + aby^2 + az^2) = \\ &s''^2 + bt''^2 + cu''^2 + bcv''^2 + abcw''^2 + acx''^2 + aby''^2 + az''^2,\end{aligned}$$

in which a, b, c, may be any values whatever.

In the paper in which Mr. Graves first publicly announced the theorem for octaves (Phil. Mag. for April, 1845), a very interesting triplet formula is deduced by aid of the new theory of imaginaries. The formula adverted to is this, viz.:

$$(ax^2+bxy+cy^2)\,(ax_{,}^2+bx_{,}y_{,}+cy_{,}^2)\,(ax_2^2+bx_2y_2+cy_2^2) = ax_3^2+bx_3y_3+cy_3^2.$$

An easy method of obtaining it, independently of that theory, has recently occurred to me; and as such independent verifications of the new doctrine, in the present stage of its progress, may not be superfluous, I may, perhaps, be permitted to offer it here, as a conclusion to this communication.

It is already known (Barlow's Theory of Numbers, p. 184), that

$$(x^2 + bxy + cy^2)\,(x_{,}^2 + bx_{,}y_{,} + cy_{,}^2) = x_0^2 + bx_0y_0 + cy_0^2,$$

in which

$$x_0 = xx_{,} - cyy_{,}, \text{ and } y_0 = xy_{,} + yx_{,} + byy_{,}.$$

Now, if instead of b, c, we write $\frac{b}{a}$, $\frac{c}{a}$, then it is plain, from the values of x_0, y_0, here exhibited, that, in order to restore the integral character of the factors and of the resulting product, we shall have to write

$$a(ax^2 + bxy + cy^2)\,(ax_{,}^2 + bx_{,}y_{,} + cy_{,}^2) = ax_0^2 + bx_0y_0 + cy_0^2;$$

similarly,

$$a(ax_0^2 + bx_0y_0 + cy_0^2)\,(ax_2^2 + bx_2y_2 + cy_2^2) = ax_3^2 + bx_3y_3 + cy_3^2;$$

consequently,

$$a^2(ax^2+bxy+cy^2)\,(ax_{,}^2+bx_{,}y_{,}+cy_{,}^2)\,(ax_2^2+bx_2y_2+cy_2^2) = ax_3^2+bx_3y_3+cy_3^2.$$

Hence, the second member must be divisible by a^2; therefore, each of the com-

ponent terms must be divisible by a^2; for no part of one term can be cancelled by a part of another, on account of the three independent factors, a, b, c; and, further, on account of these factors, $x_3 y_3$ and y_3^2 must each be divisible by a^2; therefore, y_3 itself must be divisible by a; and we see, from the first term, that x_3 must also be divisible by a. Hence, dividing each member of the equation by a^2, Mr. Graves's theorem is the result.

From the foregoing values for x_0, y_0, those for x_3, y_3, may, of course, be readily obtained: but they are already given in Mr. Graves's paper.

Belfast, August 2, 1847.

ADDITIONAL NOTE REFERRED TO IN PAGE 336.

I am indebted to the courtesy of Sir William Rowan Hamilton for the following communication, respecting the researches of John T. Graves, Esq., and for permission to append it to the foregoing Paper. It will be seen that I have had no opportunity of making any other use of it.

Note, by Professor Sir W. R. Hamilton, respecting the Researches of John T. Graves, Esq.

"You are aware, from the statement made by me to the Royal Irish Academy, on the occasion of my presenting your eight-square formula last summer, and published in the Proceedings for that evening (June 14, 1847), that my friend, John Graves, had previously sent me an equivalent formula, in a letter dated the 26th of December, 1843, which reached me before the end of that year. That letter, indeed, having been written in haste, upon a journey, contained a few errors of sign; but these were completely corrected in a shortly subsequent communication, from which the formula in the Proceedings has been transcribed. My present object is to mention that J. T. Graves, to whom I had previously communicated my theory of *quaternions*, was early led, by his extension of Euler's theorem, to conceive an analogous theory of *octaves*, involving *seven* distinct imaginaries, or square roots of negative unity, namely, *four* new roots, which he denoted by

the letters l, m, n, o, to be combined with my *three* letters, i, j, k, into one common imaginary or symbolic system. Thus, as I had already (in October and November, 1843) communicated to him and to the Academy the fundamental equations of quaternions, namely,

$$\left.\begin{array}{l} i^2 = j^2 = k^2 = -1, \\ ij = k,\ jk = i,\ ki = j, \\ ji = -k,\ kj = -i,\ ik = -j, \end{array}\right\} \tag{a}$$

which may be concisely summed up in the formula

$$i^2 = j^2 = k^2 = ijk = -1; \tag{b}$$

so he proposed to found a theory of octaves on the following equations,

$$\left.\begin{array}{l} i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1, \\ i = jk = lm = on = -kj = -ml = -no, \\ j = ki = ln = mo = -ik = -nl = -om, \\ k = ij = lo = nm = -ji = -ol = -mn, \\ l = mi = nj = ok = -im = -jn = -ko, \\ m = il = oj = kn = -li = -jo = -nk, \\ n = jl = io = mk = -lj = -oi = -km, \\ o = ni = jm = kl = -in = -mj = -lk; \end{array}\right\} \tag{c}$$

which he communicated to me, in a letter dated January 4, 1844, and which may be concisely expressed by the single but continued equation,

$$\left.\begin{array}{l} i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1 \\ = ijk = ilm = ion = jln = jmo = klo = knm. \end{array}\right\} \tag{d}$$

" In other words, as I had introduced the consideration of an *imaginary triad*, or *ternary cycle* of square roots of negative unity, namely, ijk, in which each is = the product of the two that follow it in the cyclical succession, $ijkijk$, if those two factors be taken in their order ($i = jk$, &c.), but is equal to the *negative* of that product, if the order of the two factors be reversed ($i = -kj$, &c.); so J. T. Graves extended this view to the consideration of *seven such triads*, that is to say, my triad and six new ones formed on the same type, namely,

$$ijk,\ ilm,\ ion,\ jln,\ jmo,\ klo,\ knm. \tag{e}$$

" And as I had shown that, with the equations (a) or (b), *the product of two quaternions is a quaternion*,

$$\begin{array}{l} (w + ix + jy + kz)\,(w' + ix' + jy' + kz') \\ \qquad = w'' + ix'' + jy'' + kz'', \end{array} \tag{f}$$

which satisfies the *law of the moduli,* namely,

$$(w^2 + x^2 + y^2 + z^2)\,(w'^2 + x'^2 + y'^2 + z'^2) = w''^2 + x''^2 + y''^2 + z''^2; \tag{g}$$

so my friend Graves pointed out to me, in return, that the *product of two octaves is an octave which satisfies the same modular law;* or that he could write, consistently with his extended definitions,

$$\left.\begin{aligned} &(a + ib + jc + kd + le + mf + ng + oh) \\ &\times(a' + ib' + jc' + kd' + le' + mf' + ng' + oh') \\ &= a'' + ib'' + jc'' + kd'' + le'' + mf'' + ng'' + oh''; \end{aligned}\right\} \tag{h}$$

where

$$\left.\begin{aligned} &(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \\ &\times(a'^2 + b'^2 + c'^2 + d'^2 + e'^2 + f'^2 + g'^2 + h'^2) \\ &= a''^2 + b''^2 + c''^2 + d''^2 + e''^2 + f''^2 + g''^2 + h''^2. \end{aligned}\right\} \tag{i}$$

"And thus he succeeded in connecting his eight-square formula with a theory of octaves, as I had already been led from quaternions to a four-square formula, which latter formula appears, however, to have been previously discovered by Euler.

"It was natural that, when Graves had gone so far, he should entertain the hope of extending similar principles to systems of sixteen, and generally of 2^m squares; and, accordingly, in his letter of December 26, 1843, he spoke of what he proposed to call 2^m-ions. But he soon afterwards told me that he had met with what he called 'an unexpected hitch,' in seeking to extend the law of the moduli to systems of sixteen numbers; and, in a letter of February 3, 1844, he said: 'I cannot help harping on the strangeness of not being able to arrange the product of two sums of sixteen squares as a sum of sixteen rational squares.' He then proceeded to point out certain *cases* in which this arrangement could be effected, and enclosed me two *schemes* with that view; and, after offering some suggestions respecting the effects of signs and substitutions, he said, 'it ought to be capable of *à priori* proof that the problem is impossible, *if it be so.*' He also expressed a wish that I should attempt to furnish a proof of its impossibility. Being engaged at the time on other matters, I forbore to make that attempt; and as I believe that my friend Graves did not pursue the inquiry, the honour of the *à priori* investigation respecting the products of sums of sixteen squares has been reserved for you.

"I regret that you did not apply to me to furnish you sooner with a sketch of those early researches of my friend, John Graves, which contained other things that would have interested you; for instance, a mode of introducing certain arbitrary coefficients into the eight-square formula, and certain extensions of results from squares to

binary products. I hope that he may yet be induced to furnish an account of them to the Philosophical Magazine. The present hasty and imperfect sketch will not come to your hands till after the last pages of your Paper have passed through those of the printers; indeed, as you have authorized me, by anticipation, to send it to press as a concluding note to your essay, and as it is desired by the Council of the Academy to publish the Second Part of the Twenty-first Volume of the Transactions forthwith, you will, perhaps, see it first in print, and will, at all events, have no opportunity of incorporating, on the present occasion, any remarks respecting it with your own very interesting communication.

"W. R. H.

"OBSERVATORY, *March* 7, 1848.

"*To Professor Young.*"